

# Weak Hopf algebras and the distribution of involutions in symmetric groups

Takahiro Hayashi

Graduate School of Mathematics, Nagoya University,  
Chikusa-ku, Nagoya 464-8602, Japan  
E-mail: hayashi@math.nagoya-u.ac.jp

## Abstract

By computing Frobenius-Schur indicators of modules of certain weak Hopf algebras, we give a formula for the number of involutions in symmetric groups, which are contained in a given coset with respect to a given Young subgroup.

## 1 Introduction

Let  $\mathfrak{S}_n$  be the symmetric group of  $n$ -letters. Then we have the following classical identity in combinatorial representation theory:

$$|\{a \in \mathfrak{S}_n \mid a^2 = 1\}| = |\text{STab}(n)|, \quad (1.1)$$

where  $\text{STab}(n)$  denotes the set of all standard tableaux of size  $n$ . Besides a proof based on RSK correspondence, there is a proof of this identity, which is based on Frobenius-Schur indicators. Let  $G$  be an arbitrary finite group. Then the  $r$ -th root number function  $R_G^r(a) := |\{c \in G \mid c^r = a\}|$  is given by

$$R_G^r = \sum_{\chi \in \text{Irr } G} \text{FS}_r(\chi) \chi, \quad (1.2)$$

where  $\text{Irr } G$  denotes the set of (complex) irreducible characters of  $G$  and  $\text{FS}_r(\chi)$  denotes the  $r$ -th Frobenius-Schur indicator of  $\chi$ . Hence (1.1) follows from  $\text{FS}_2(\chi) = 1$  ( $\chi \in \text{Irr } \mathfrak{S}_n$ ) and  $\sum_{\chi \in \text{Irr } \mathfrak{S}_n} \chi(1) = |\text{STab}(n)|$ .

Let  $H$  be a subgroup of  $G$  and let  $b$  be an element of  $G$ . In this paper, we consider the following *coset-wise root number function*:

$$R_{G,bH}^r(a) := |\{c \in bH \mid c^r = a\}|.$$

The support  $K$  of the restricted function  $R_{G,bH}^r|_H$  becomes a subgroup of  $H$  and the restriction of  $R_{G,bH}^r$  on  $K$  has the expansion

$$R_{G,bH}^r|_K = \sum_{\chi \in \text{Irr } K} \text{FS}_r(L_\chi) \chi,$$

where  $\text{FS}_r(L_\chi)$  denotes the  $r$ -th Frobenius-Schur indicator of certain simple module  $L_\chi$  of a weak Hopf algebra (WHA)  $\mathcal{F}(G, X)$  attached to  $G$  and  $X := G/H$ . When  $G = \mathfrak{S}_n$  and  $H = \mathfrak{S}_{n-1}$ , we give an explicit formulas of  $\text{FS}_r(L_\chi)$  and  $R_{G,bH}^r$  for every  $r > 0$ . When  $G = \mathfrak{S}_n$  and  $H$  is a Young subgroup  $\mathfrak{S}_\alpha$ , we determine the value  $\text{FS}_2(L_\chi)$  and give an explicit formula for the number of involutions in  $b\mathfrak{S}_\alpha$ . As a special case, we obtain

$$|\{a \in b\mathfrak{S}_m \mid a^2 = 1\}| = \begin{cases} |\text{STab}(m-k)| & (\mathfrak{S}_m b \mathfrak{S}_m = \mathfrak{S}_m b^{-1} \mathfrak{S}_m) \\ 0 & (\mathfrak{S}_m b \mathfrak{S}_m \neq \mathfrak{S}_m b^{-1} \mathfrak{S}_m), \end{cases} \quad (1.3)$$

for each  $0 < m < n$  and  $b \in \mathfrak{S}_n$ , where  $k := |\{1, 2, \dots, m\} \setminus b\{1, 2, \dots, m\}|$ . Also, we obtain

$$|\{a \in b(\mathfrak{S}_m \times \mathfrak{S}_{m'}) \mid a^2 = 1\}| = k! |\text{STab}(m-k)| |\text{STab}(m'-k)|, \quad (1.4)$$

where  $m' := n - m$ .

By counting  $R_{\mathfrak{S}_n}^2(1)$  using (1.4), we obtain

$$|\text{STab}(n)| = \sum_{0 \leq k \leq \min\{m, m'\}} \frac{m! m'!}{k! (m-k)! (m'-k)!} |\text{STab}(m-k)| |\text{STab}(m'-k)|. \quad (1.5)$$

In [15], Ng and Schauenburg defined Frobenius-Schur indicators as invariants of (objects of) pivotal fusion categories (See also [6]). Also, Schauenburg [17, 18,

19] gave several results for Frobenius-Schur indicators of group-theoretical fusion categories. Since the representation category of  $\mathcal{F}(G, X)$  is group-theoretical by Andruskiewitsch-Natale [1] and Mombelli-Natale [13], our general results for Frobenius-Schur indicators of  $\mathcal{F}(G, X)$ -modules overlap with Schauenburg's results significantly. Nevertheless, we give WHA counterparts of his results, since WHA approach seems to be more elementary than his category-theoretic approach. In fact, our approach clarifies the importance of the  $G$ -sets  $X$  and  $X \times X$ , which did not play important roles in his paper.

The outline of the paper is as follows. In Section 2 we give the definition of the algebra  $\mathcal{F}(G, X)$ . In Section 3, we define and study Frobenius-Schur indicators of  $\mathcal{F}(G, X)$ -modules. In Section 4 we give a relation between second indicators of  $\mathcal{F}(G, X)$ -modules and Kawanaka-Matsuyama indicators [10] of  $\mathbb{C}K$ -modules. In Section 5 and Section 6, we compute second indicators of  $\mathcal{F}(\mathfrak{S}_n, \mathfrak{S}_\alpha)$ -modules and indicators of  $\mathcal{F}(\mathfrak{S}_n, \mathfrak{S}_{n-1})$ -modules, respectively. Also, we give the corresponding results for  $R_{G,bH}^r$  in these sections. In Section 7 we give a correspondence between invariant bilinear forms on  $\mathcal{F}(G, X)$ -modules and invariant bilinear pairings of some  $\mathbb{C}K$ -modules. In Section 8 we verify that our definition of Frobenius-Schur indicators of  $\mathcal{F}(G, X)$ -modules coincides with that of Ng-Schauenburg [15].

The author thank K. Shimizu for telling him about the Frobenius-Schur indicators.

## 2 Preliminaries

Throughout this paper, all modules are assumed to be finite dimensional over the complex number field  $\mathbb{C}$ . Let  $G$  be a finite group and let  $X$  be a finite left  $G$ -set. For  $x \in X$ , we denote by  $G_x$  the *stabilizer* of  $G$  at  $x$ , that is,  $G_x = \{a \in G \mid ax = x\}$ . Let  $\mathbb{C}^X \rtimes G$  be the  $\mathbb{C}$ -linear span of the symbols  $e_x a$  ( $a \in G, x \in X$ ). Then  $\mathbb{C}^X \rtimes G$  becomes an algebra via

$$(e_x a)(e_y b) = \delta_{x,ay} e_x ab.$$

By identifying  $a \in G$  with  $\sum_{x \in X} e_x a$ ,  $\mathbb{C}G$  becomes a subalgebra of  $\mathbb{C}^X \rtimes G$ . The elements  $e_x := e_x 1_G$  ( $x \in X$ ) are mutually orthogonal idempotents and give a partition of unity of  $\mathbb{C}^X \rtimes G$ .

Let  $M$  be a left  $\mathbb{C}^X \rtimes G$ -module and let  $\Omega$  be an orbit of the  $G$ -set  $X$ . We say that  $M$  is of *type*  $\Omega$  if  $M = \bigoplus_{x \in \Omega} e_x M$ . We note that each  $\mathbb{C}^X \rtimes G$ -module has a unique decomposition  $M = \sum_{\Omega \in G \backslash X} M_\Omega$  such that  $M_\Omega$  is of type  $\Omega$ .

It seems that the following is a folklore among some communities of Hopf algebraists.

**Proposition 2.1** (cf. [12] page 241, [11] Lemma 3.2, Theorem 3.3) (1) *Let  $\Omega = Gx$  be an orbit of  $X$  and let  $V$  be a left  $\mathbb{C}G_x$ -module. Then  $\mathcal{I}_x(V) := \mathbb{C}G \otimes_{\mathbb{C}G_x} V$  becomes a  $\mathbb{C}^X \rtimes G$ -modules via*

$$a(b \otimes v) = ab \otimes v, \quad e_y(b \otimes v) = \delta_{y,bx} b \otimes v \quad (a, b \in G, y \in X, v \in V).$$

(2) *The correspondence  $\mathcal{I}_x$  gives an equivalence between the category of  $\mathbb{C}G_x$ -modules and the category of  $\mathbb{C}^X \rtimes G$ -modules of type  $\Omega$ .*

Let  $\mathcal{F} = \mathcal{F}(G, X)$  be the  $\mathbb{C}$ -linear span of the symbols  $e_y^x a$  ( $a \in G, x, y \in X$ ). Then  $\mathcal{F}$  becomes an algebra via

$$(e_y^x a)(e_w^z b) = \delta_{x,az} \delta_{y,aw} e_y^x ab.$$

Let  $\Delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$  and  $\varepsilon : \mathcal{F} \rightarrow \mathbb{C}$  be linear maps given by

$$\Delta(e_y^x a) = \sum_{z \in X} e_z^x a \otimes e_y^z a, \quad \varepsilon(e_y^x a) = \delta_{xy}.$$

Then  $\mathcal{F}$  becomes a  $X$ -face algebra with antipode  $S : \mathcal{F} \rightarrow \mathcal{F}; e_y^x a \mapsto a^{-1} e_x^y$  (cf. [7]). Hence  $\mathcal{F}$  is a weak Hopf algebra (cf. [2]). We call  $\mathcal{F}(G, X)$  the *group-like face algebra* of  $(G, X)$ .

Let  $\Omega = G(x, y)$  be an *orbital* of  $X$ , that is,  $\Omega \in G \backslash (X \times X)$ . Since  $\mathcal{F}(G, X) \cong \mathbb{C}^{X \times X} \rtimes G$  as algebras, we have an equivalence  $\mathcal{I}_{xy}$  between the category of  $\mathbb{C}G_{xy}$ -modules and the category of  $\mathcal{F}(G, X)$ -modules of type  $\Omega$ , where  $G_{xy}$  stands for the *two-point stabilizer*  $G_x \cap G_y$ . In particular, if  $\{V(\lambda)\}$

is a set of representatives for the isomorphism classes of simple  $\mathbb{C}G_{xy}$ -modules, then  $\{\mathcal{I}_{xy}(V(\lambda))\}$  is a set of representatives for the isomorphism classes of simple  $\mathcal{F}(G, X)$ -modules of type  $\Omega$ .

### 3 Frobenius-Schur indicators

We define elements  $\int^{[r]}$  ( $r \geq 1$ ) of  $\mathcal{F}(G, X)$  by  $\int^{[1]} = \int := \frac{1}{|G|} \sum_{a \in G} \sum_{x \in X} e_x^x a$  and  $\int^{[r]} := (m^{(r)} \circ \Delta^{(r)})(\int)$  ( $r \geq 2$ ) respectively, where  $m^{(r)} : \mathcal{F}(G, X)^{\otimes r} \rightarrow \mathcal{F}(G, X)$  and  $\Delta^{(r)} : \mathcal{F}(G, X) \rightarrow \mathcal{F}(G, X)^{\otimes r}$  denote the iterations of the product and the coproduct of  $\mathcal{F}(G, X)$  respectively, that is,  $m^{(3)}(\alpha, \beta, \gamma) = \alpha\beta\gamma$  and  $\Delta^{(3)}(\alpha) = (\Delta \otimes \text{id})(\Delta(\alpha))$ , for example. Then,  $\int$  is an idempotent two-sided integral of  $\mathcal{F}(G, X)$  (cf. [2]), that is,  $\int^2 = \int$  and

$$\alpha \int = \varepsilon^L(\alpha) \int, \quad \int \alpha = \int \varepsilon^R(\alpha) \quad (\alpha \in \mathcal{F}(G, X)), \quad (3.1)$$

where,

$$\varepsilon^L(\alpha) = \sum_{x, y, z \in X} \varepsilon(e_z^x \alpha) e_y^z, \quad \varepsilon^R(\alpha) = \sum_{x, y, z \in X} e_z^x \varepsilon(\alpha e_y^z). \quad (3.2)$$

Let  $M$  be a finite-dimensional  $\mathcal{F}(G, X)$ -module. We define the  $r$ -th *Frobenius-Schur indicator* of  $M$  by  $\text{FS}_r(M) := \text{Tr}_M(\int^{[r]})$ .

**Lemma 3.1** (1) *Explicitly, the elements  $\int^{[r]}$  are given by*

$$\int^{[r]} = \frac{1}{|G|} \sum_{a \in G} \sum_{x \in X} \delta_{x, a^r x} e_{a^{-1}x}^x a^r = \frac{1}{|G|} \sum_{a \in G} \sum_{x \in X} \delta_{x, a^r x} a^r e_{a^{-1}x}^x. \quad (3.3)$$

(2) *The element  $\int^{[r]}$  is central.*

*Proof.* Part (1) follows from the following computations:

$$\begin{aligned}
& (m^{(r)} \circ \Delta^{(r)})(e_x^x a) \\
&= \sum_{y_1 \in X} \cdots \sum_{y_{r-1} \in X} (e_{y_1}^x a) (e_{y_2}^{y_1} a) \cdots (e_{y_{r-1}}^{y_{r-2}} a) (e_x^{y_{r-1}} a) \\
&= \sum_{y_1 \in X} \cdots \sum_{y_{r-1} \in X} (e_{y_1}^x a) (e_{y_2}^{y_1} a) \cdots (e_{y_{r-2}}^{y_{r-3}} a) \delta_{y_{r-2}, a y_{r-1}} \delta_{y_{r-1}, a x} (e_{y_{r-1}}^{y_{r-2}} a^2) \\
&= \sum_{y_1 \in X} \cdots \sum_{y_{r-3} \in X} (e_{y_1}^x a) (e_{y_2}^{y_1} a) \cdots (e_{a^2 x}^{y_{r-3}} a) (e_{a x}^{a^2 x} a^2) \\
&= \cdots \\
&= (e_{a^{r-1} x}^x a) (e_{a^{r-2} x}^{a^{r-1} x} a^{r-1}) = \delta_{x, a^r x} e_{a^{-1} x}^x a^r = \delta_{x, a^r x} a^r e_{a^{-1} x}^x.
\end{aligned}$$

Let  $c$  be an element of  $G$ . Replacing  $a$  and  $x$  in (3.3) by  $cbc^{-1}$  and  $cy$  respectively, we obtain

$$\begin{aligned}
\int^{[r]} c &= \left( \frac{1}{|G|} \sum_{b \in G} \sum_{y \in X} \delta_{cy, cb^r y} e_{cb^{-1}y}^{cy} cb^r c^{-1} \right) c \\
&= \frac{1}{|G|} \sum_{b \in G} \sum_{y \in X} \delta_{y, b^r y} c e_{b^{-1}y}^y b^r = c \int^{[r]}.
\end{aligned}$$

For  $y, z \in X$ , we have

$$e_z^y \int^{[r]} = \frac{1}{|G|} \sum_{a \in G} \delta_{y, a^r y} \delta_{z, a^{-1} y} e_{a^{-1} y}^y a^r = \frac{1}{|G|} \sum_{a \in G} \delta_{y, a^r y} \delta_{z, a^{-1} y} a^r e_{a^{-1} y}^y = \int^{[r]} e_z^y.$$

Since  $g$ 's and  $e_z^y$ 's generate the algebra  $\mathcal{F}(G, X)$ , this proves Part (2).  $\square$

**Theorem 3.2** (cf. Schauenburg [17], Theorem 4.1) *For each  $x, y \in X$  and  $\mathbb{C}G_{xy}$ -module  $V$ , we have*

$$\text{FS}_r(\mathcal{I}_{xy}(V)) = \frac{1}{|G_{xy}|} \sum_{a \in G[x, y; r]} \text{Tr}_V(a^{-r}), \quad (3.4)$$

where  $G[x, y; r] := \{a \in G \mid ax = y, a^r x = x\}$ .

*Proof.* We first note that the right-hand side of (3.4) is well-defined, since  $a^{-r} \in K := G_{xy}$  for each  $a \in G[x, y; r]$ . Also, we note that we may assume that  $V$  is a simple  $\mathbb{C}K$ -module, since both the right-hand side and the left-hand side

of (3.4) are additive with respect to  $V$ . Then, by Proposition 2.1 (2),  $\mathcal{I}_{xy}(V)$  is a simple  $\mathcal{F}(G, X)$ -module. Hence by Schur's lemma, the action of the central element  $\int^{[r]}$  on  $\mathcal{I}_{xy}(V)$  is given by some scalar. Therefore, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{I}_{xy}(V)}(\int^{[r]}) &= \frac{\dim \mathcal{I}_{xy}(V)}{\dim(\mathbb{C}K \otimes_{\mathbb{C}K} V)} \mathrm{Tr}_{\mathbb{C}K \otimes_{\mathbb{C}K} V}(\int^{[r]}) \\ &= \frac{|G|}{|K|} \mathrm{Tr}_{\mathbb{C}K \otimes_{\mathbb{C}K} V}(\int^{[r]}).\end{aligned}$$

By Lemma 3.1 (1), we have

$$\begin{aligned}\int^{[r]}(1_G \otimes v) &= \frac{1}{|G|} \sum_{a,z} \delta_{z,a^r z} \delta_{z,x} \delta_{a^{-1}z,y} a^r \otimes v \\ &= \frac{1}{|G|} \sum_{c \in G[x,y;r]} 1_G \otimes c^{-r} v.\end{aligned}$$

Therefore,

$$\mathrm{Tr}_{\mathbb{C}K \otimes_{\mathbb{C}K} V}(\int^{[r]}) = \frac{1}{|G|} \sum_{c \in G[x,y;r]} \mathrm{Tr}_V(c^{-r}).$$

This proves (3.4).  $\square$

Let  $H$  be a subgroup of  $G$ . We define the  $r$ -th root number function  $R_G^r$  and the  $r$ -th coset-wise root number function  $R_{G,bH}^r$  by

$$\begin{aligned}R_G^r(a) &= |\{c \in G \mid c^r = a\}|, \\ R_{G,bH}^r(a) &= |\{c \in bH \mid c^r = a\}|,\end{aligned}$$

respectively. We note that  $R_G^r$  is a class function and that

$$R_{G,hbH}^r(a) = R_{G,bH}^r(h^{-1}ah) \quad (3.5)$$

for each  $a, b \in G$  and  $h \in H$ . In particular, we have

$$R_{G,hbH}^r(1) = R_{G,bH}^r(1). \quad (3.6)$$

By (3.6), the assignment  $HbH \mapsto R_{G,HbH}^r(1) := R_{G,bH}^r(1)$  gives a well-defined function on  $H \backslash G/H$ .

**Proposition 3.3** *The root number function satisfy*

$$R_G^r(1) = \sum_{HbH \in H \backslash G/H} \frac{|H|}{|H \cap bHb^{-1}|} R_{G,HbH}^r(1). \quad (3.7)$$

*Proof.* By definition, we have

$$\begin{aligned} R_G^r(1) &= \sum_{bH \in G/H} R_{G,bH}^r(1) \\ &= \sum_{Hb_1H \in H \backslash G/H} c_{Hb_1H} R_{G,Hb_1H}^r(1), \end{aligned}$$

where  $c_{Hb_1H} := |\{bH \in G/H \mid HbH = Hb_1H\}|$ . Since  $c_{Hb_1H}$  is equals to the size of the  $H$ -orbit through  $y := b_1H \in X := G/H$ , it equals  $|H|/|H_y| = |H|/|H \cap b_1Hb_1^{-1}|$ . This proves (3.7).  $\square$

**Theorem 3.4** (cf. Schauenburg [17], Lemma 4.5) *For each  $a \in H$  and  $y = bH \in X := G/H$ , we have*

$$\begin{aligned} & \left| \{c \in bH \mid c^r = a\} \right| \\ &= \begin{cases} \sum_{\lambda} \text{FS}_r(\mathcal{I}_{xy}(V(\lambda))) \chi_{\lambda}(a) & (ay = y) \\ 0 & (ay \neq y), \end{cases} \end{aligned} \quad (3.8)$$

where  $x = H \in X$ ,  $\{V(\lambda)\}$  is as in Section 2 and  $\chi_{\lambda} = \text{Tr}_{V(\lambda)}$  denotes the character of  $V(\lambda)$ .

*Proof.* To begin with, we show that the left-hand side of (3.8) is non-zero only if  $ay = y$ . Suppose that  $c^r = a$  for some  $c \in bH$ . Since  $c \in bH$ , we have  $cx = bx = y$ . Hence

$$ay = c^r y = c^{r+1} x = cax = cx = y.$$

Let  $K$  be the two-point stabilizer  $G_{xy}$ . By (3.5),  $R_{G,bH}^r|_K$  is a class function on  $K$ . Hence

$$R_{G,bH}^r|_K = \sum_{\lambda} (R_{G,bH}^r|_{\chi_{\lambda}})_K \chi_{\lambda},$$

where  $(|)_K$  denotes the usual inner product of the space of class functions on  $K$ , that is,  $(f|g)_K := |K|^{-1} \sum_{a \in K} f(a) \overline{g(a)}$ . Therefore, it suffices to show that  $(R_{G,bH}^r|_{\chi_{\lambda}})_K = \text{FS}_r(\mathcal{I}_{xy}(V)(\lambda))$ . By definition, we have

$$\begin{aligned} (R_{G,bH}^r|_{\chi_{\lambda}})_K &= \frac{1}{|K|} \sum_{a \in K} |\{c \in bH \mid c^r = a\}| \chi_{\lambda}(a^{-1}) \\ &= \frac{1}{|K|} \sum_{c \in bH; c^r \in K} \chi_{\lambda}(c^{-r}). \end{aligned} \quad (3.9)$$



Since

$$\{c \in bH \mid c^r \in K\} = \{c \in G \mid cx = y, c^r y = y\} = G[x, y; r],$$

the right-hand side of (3.9) coincides with that of (3.4) for  $V = V(\lambda)$ .  $\square$

## 4 Twisted Frobenius-Schur indicators

Let  $K$  be a finite group. Let  $\phi$  be an automorphism of  $K$  and let  $k_0$  be an element of  $K$ . We say that  $(\phi, k_0)$  is an *outer involution* of  $K$  if  $\phi^2(k) = k_0^{-1}kk_0$  ( $k \in K$ ) and  $\phi(k_0) = k_0$ . We note that if  $K \leq G$  and  $t \in G$  satisfies  $t^{-1}Kt = K$  and  $t^2 \in K$ , then  $((-)^t, t^2)$  is an outer involution of  $K$ , where  $(-)^t : K \rightarrow K; k \rightarrow t^{-1}kt$ . Conversely, for an outer involution  $(\phi, k_0)$ , there exists a group  $G \geq K$  and  $t \in G \setminus K$  such that  $(\phi, k_0) = ((-)^t, t^2)$ . Explicitly,  $G$  is given by  $G = K \amalg tK$ , which is equipped with product  $(tk)(tk') = k_0(\phi(k)k')$ ,  $(tk)k' = t(kk')$ ,  $k(tk') = t(\phi(k)k')$  ( $k, k' \in K$ ), where  $tK = \{tk \mid k \in K\}$  is a copy of the set  $K$ .

Let  $V$  be a finite-dimensional  $\mathbb{C}K$ -module. We define  $(\phi, k_0)$ -*twisted second Frobenius-Schur indicator* of  $V$  by

$$\text{FS}_2(V, \phi, k_0) = \frac{1}{|K|} \sum_{k \in K} \text{Tr}_V(k_0 \phi(k)k).$$

When  $(\phi, k_0) = ((-)^t, t^2)$ , we write  $\text{FS}_2(V, t) = \text{FS}_2(V, \phi, k_0)$ . It agrees with Kawanaka-Matsuyama's indicator (cf. [10]), that is,

$$\text{FS}_2(V, t) = \frac{1}{|K|} \sum_{k \in K} \text{Tr}_V((tk)^2).$$

We say that an orbital  $\Omega$  is *symmetric* (or *self-paired*) if  $\Omega^\top = \Omega$ , where  $\Omega^\top := \{(y, x) \mid (x, y) \in \Omega\}$ .

**Proposition 4.1** (1) *An orbital  $\Omega = G(x, y)$  is symmetric if and only if there exists an element  $t \in G$  such that  $tx = y$ ,  $ty = x$ . In this case,  $K := G_{xy}$  satisfies  $t^{-1}Kt = K$  and  $t^2 \in K$ .*

(2) *Let  $H$  be a subgroup of  $G$  and let  $b$  be an element of  $G$ . Let  $x_0$  be the element*

$H$  of  $X = G/H$  and let  $\Omega$  be  $G(x_0, bx_0)$ . Then  $\Omega$  is symmetric if and only if  $HbH = Hb^{-1}H$ .

*Proof.* Part (1) is obvious. Since there exists a bijection  $H \backslash G/H \cong G \backslash (X \times X)$ ;  $HbH \mapsto G(x_0, bx_0)$  (cf. [4] p240),  $G(x_0, bx_0)$  is symmetric if and only if  $HbH = Hb^{-1}H$ .  $\square$

**Theorem 4.2** (cf. Schauenburg [19], Proposition 3.2) *Let  $V$  be a  $\mathbb{C}G_{xy}$ -module.*

- (1) *If  $\Omega = G(x, y)$  is not symmetric, then  $\text{FS}_2(\mathcal{I}_{x,y}(V)) = 0$ .*
- (2) *Suppose that  $\Omega$  is symmetric and that  $t \in G$  satisfies  $t(x, y) = (y, x)$ . Then,*

$$\text{FS}_2(\mathcal{I}_{x,y}(V)) = \text{FS}_2(V, t).$$

*Proof.* Since  $G[x, y; 2] = \{a \in G \mid ax = y, ay = x\}$  is empty if  $\Omega^\top \neq \Omega$ , Part (1) follows from (3.4). If  $t \in G$  satisfies  $t(x, y) = (y, x)$ , then we have  $G[x, y; 2] = G_{xy} t^{-1}$ . Hence Part (2) also follows from (3.4).  $\square$

## 5 Symmetric groups I

For each subset  $S$  of  $[n] := \{1, 2, \dots, n\}$ , we define a subgroup  $\mathfrak{S}(S) \cong \mathfrak{S}_{|S|}$  of  $G := \mathfrak{S}_n$  by  $\mathfrak{S}(S) := \{a \in \mathfrak{S}_n \mid ai = i \ (i \in [n] \setminus S)\}$ . For a set  $S \subset \mathbb{Z}$  and an integer  $\epsilon$ , we set  $\epsilon + S = \{\epsilon + s \mid s \in S\}$ . Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  be a sequence of positive integers such that  $\alpha_1 + \dots + \alpha_\ell = n$ . Let  $\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \dots \times \mathfrak{S}_{\alpha_\ell}$  be the corresponding Young subgroup of  $\mathfrak{S}_n$ . Here, we identify  $\mathfrak{S}_\alpha$  with  $\mathfrak{S}(A_1) \cdots \mathfrak{S}(A_\ell)$  as usual, where

$$A_1 = [\alpha_1], A_2 = \alpha_1 + [\alpha_2], \dots, A_\ell = \alpha_1 + \dots + \alpha_{\ell-1} + [\alpha_\ell]. \quad (5.1)$$

Next, define a set  $X = \binom{[n]}{\alpha}$  by

$$X := \left\{ B = (B_1, \dots, B_\ell) \in (2^{[n]})^\ell \mid [n] = \coprod_i B_i, \quad |B_i| = \alpha_i \quad (1 \leq i \leq \ell) \right\}.$$

Then  $X$  becomes a transitive  $G$ -set via  $a(B_1, \dots, B_\ell) := (a(B_1), \dots, a(B_\ell))$ . Since the stabilizer  $G_A$  of  $G$  at  $A := (A_1, \dots, A_\ell)$  is  $\mathfrak{S}_\alpha$ ,  $\mathfrak{S}_n / \mathfrak{S}_\alpha \cong X$ ;  $b\mathfrak{S}_\alpha \mapsto bA$

as  $G$ -sets. It is known that  $G(B, C) \mapsto [|B_i \cap C_j|]_{ij}$  gives a bijection from  $G \backslash (X \times X)$  onto

$$M_\alpha := \{ \Gamma = [\gamma_{ij}]_{ij} \in \text{Mat}(\ell, \mathbb{Z}_{\geq 0}) \mid \sum_i \gamma_{ij} = \alpha_j = \sum_i \gamma_{ji} \quad (1 \leq j \leq \ell) \}.$$

See e.g. [9]. Note that  $G(B, C)$  is a symmetric orbital if and only if  $[|B_i \cap C_j|]_{ij}$  is a symmetric matrix.

Let  $B = (B_1, \dots, B_\ell) = bA$  be an element of  $X$ . For  $1 \leq i, j \leq \ell$ , we set  $B_{ij} := A_i \cap B_j$  and  $\gamma_{ij} := |B_{ij}|$ . Also, we set  $A_{ij} = \epsilon_{ij} + [\gamma_{ij}]$ , where  $\epsilon_{11} = 0$ ,  $\epsilon_{12} = \gamma_{11}, \dots$ ,  $\epsilon_{1\ell} = \gamma_{11} + \dots + \gamma_{1,\ell-1}$ ,  $\epsilon_{21} = \gamma_{11} + \dots + \gamma_{1\ell} = \alpha_1$ ,  $\epsilon_{22} = \alpha_1 + \gamma_{21}$ ,  $\dots$ ,  $\epsilon_{ij} = \alpha_1 + \dots + \alpha_{i-1} + \gamma_{i1} + \dots + \gamma_{i,j-1}, \dots$ . By definition, we have  $A_i = A_{i1} \coprod \dots \coprod A_{i\ell}$  for each  $1 \leq i \leq \ell$ . For each  $i, j$ , we fix a bijection  $u_{ij} : A_{ij} \cong B_{ij}$  and define  $u \in \mathfrak{S}_n$  by  $u|_{A_{ij}} = u_{ij}$ . Let  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\ell)$  be the sequence  $(\gamma_{11}, \gamma_{12}, \dots, \gamma_{1\ell}, \gamma_{21}, \dots, \gamma_{2\ell}, \dots, \gamma_{\ell 1}, \dots, \gamma_{\ell \ell})$  and let  $K_0$  be the subgroup  $\mathfrak{S}_\gamma = \mathfrak{S}_{\gamma_1} \times \dots \times \mathfrak{S}_{\gamma_\ell}$  of  $\mathfrak{S}_n$ , where  $\mathfrak{S}_0 = \{1\}$ . Note that we have  $a(\epsilon_{ij} + s) = \epsilon_{ij} + a_{ij}s$  for  $a = (a_{11}, \dots, a_{1\ell}, \dots, a_{\ell 1}, \dots, a_{\ell \ell}) \in \mathfrak{S}_\gamma$  and  $s \in [\gamma_{ij}]$ . Let  $K$  the two-point stabilizer  $G_{AB}$ .

**Lemma 5.1** (1)  $K = \prod_{ij} \mathfrak{S}(B_{ij})$ .

(2) The correspondence  $k \mapsto u^{-1}ku$  gives a group isomorphism  $\psi : K \cong K_0$ .

*Proof.* (1) For each  $a \in \mathfrak{S}_n$ ,  $a \in K$  if and only if  $aA_i = A_i$  and  $aB_j = B_j$  for all  $i, j$ , if and only if  $aB_{ij} = B_{ij}$  for all  $i, j$ . Hence, we have Part (1).

(2) Since  $K_0 = \prod_{ij} \mathfrak{S}(A_{ij})$  and  $|\mathfrak{S}(A_{ij})| = \gamma_{ij}! = |\mathfrak{S}(B_{ij})|$ , it suffices to show that  $uku^{-1} \in \mathfrak{S}(B_{ij})$  for each  $k \in \mathfrak{S}(A_{ij})$ . Let  $s$  be an element of  $[n] \setminus B_{ij}$ . Since  $u^{-1}s \in [n] \setminus A_{ij}$ , we have  $ku^{-1}s = u^{-1}s$ . This proves the assertion.  $\square$

Now suppose that  $G(A, B)$  is a symmetric orbital. We define  $t_0, t \in \mathfrak{S}_n$  by  $t_0(\epsilon_{ij} + s) = \epsilon_{ji} + s$  ( $s \in [\gamma_{ij}]$ ) and  $t = ut_0u^{-1}$ . Then we have  $t_0^2 = 1_G$ ,  $t_0|_{A_{ii}} = \text{id}$  and  $t_0(A_{ij}) = A_{ji}$ . Moreover,  $t_0at_0 = a^\top$  for  $a = (a_{11}, \dots, a_{1\ell}, \dots, a_{\ell 1}, \dots, a_{\ell \ell}) \in K_0$ , where

$$a^\top := (a_{11}, a_{21}, \dots, a_{\ell 1}, \dots, a_{1\ell}, \dots, a_{\ell \ell}).$$

Since  $t(B_{ij}) = B_{ji}$ , we have  $tA = B$ ,  $tB = A$ . Moreover, we have  $t\psi^{-1}(a)t = \psi^{-1}(a^\top)$  for each  $a \in K_0$ .

For  $m \geq 0$ , let  $\mathcal{P}(m)$  be the set of partitions of  $m$  and let  $\{V(\lambda) \mid \lambda \in \mathcal{P}(m)\}$  be a complete representatives of simple  $\mathbb{C}\mathfrak{S}_m$ -modules such that  $\dim V(\lambda) = |\text{STab}(\lambda)|$ , where  $\text{STab}(\lambda)$  denotes the set of standard tableaux of shape  $\lambda$ . We denote the character of  $V(\lambda)$  by  $\chi_\lambda$ . Note that  $\mathcal{P}(0)$  is a single element set  $\{()\}$  and that  $V(())$  is a one-dimensional module of  $\mathfrak{S}_0 = \{1\}$ .

Let  $\mathcal{P}(\Gamma)$  be the set of matrices  $\Lambda = [\lambda_{ij}]_{1 \leq i, j \leq \ell}$  of partitions such that  $\lambda_{ij} \in \mathcal{P}(\gamma_{ij})$ . For each  $\Lambda = [\lambda_{ij}] \in \mathcal{P}(\Gamma)$ , define a simple  $\mathbb{C}K_0$ -module  $V(\Lambda)$  by the following outer tensor product:

$$V(\Lambda) = V(\lambda_{11}) \boxtimes V(\lambda_{12}) \boxtimes \cdots \boxtimes V(\lambda_{1\ell}) \boxtimes \cdots \boxtimes V(\lambda_{\ell 1}) \boxtimes \cdots \boxtimes V(\lambda_{\ell\ell}).$$

Then  $\{V(\Lambda) \mid \Lambda \in \mathcal{P}(\Gamma)\}$  gives a complete representative of simple  $\mathbb{C}K_0$ -modules. Hence  $\{V(\Lambda)^\psi \mid \Lambda \in \mathcal{P}(\Gamma)\}$  and  $\{\mathcal{I}_{A,B}(V(\Lambda)^\psi) \mid \Lambda \in \mathcal{P}(\Gamma)\}$  give complete representatives of simple  $\mathbb{C}K$ -modules and simple  $\mathcal{F}(G, X)$ -modules of type  $G(A, B)$ , respectively. Here, the action of  $K$  on  $V(\Lambda)^\psi := V(\Lambda)$  is given by  $(k, v) \mapsto \psi(k)v$  ( $k \in K$ ,  $v \in V(\Lambda)^\psi$ ).

**Theorem 5.2** *For each  $\Lambda \in \mathcal{P}(\Gamma)$ , the second Frobenius-Schur indicator of the  $\mathcal{F}(G, X)$ -module  $\mathcal{I}_{A,B}(V(\Lambda)^\psi)$  is given by*

$$\text{FS}_2(\mathcal{I}_{A,B}(V(\Lambda)^\psi)) = \begin{cases} 1 & (\Gamma^\top = \Gamma \text{ and } \Lambda^\top = \Lambda) \\ 0 & (\Gamma^\top \neq \Gamma \text{ or } \Lambda^\top \neq \Lambda). \end{cases} \quad (5.2)$$

*Proof.* By Theorem 4.2 (1), we may assume  $\Gamma^\top = \Gamma$ . Hence

$$\text{FS}_2(\mathcal{I}_{A,B}(V(\Lambda)^\psi)) = \text{FS}_2(V(\Lambda)^\psi, t) = \frac{1}{|K|} \sum_{k \in K} \text{Tr}_{V(\Lambda)}(\psi((tk)^2))$$

by Theorem 4.2 (2). Since  $\psi((tk)^2) = t_0 a t_0 a = a^\top a$  for  $k = \psi^{-1}(a)$  and  $a \in K_0$ , the right-hand side equals

$$\begin{aligned} & \frac{1}{|K_0|} \sum_{(a_{11}, \dots, a_{\ell\ell}) \in K_0} \text{Tr}_{V(\Lambda)}(a_{11}a_{11}, a_{21}a_{12}, \dots, a_{\ell 1}a_{1\ell}, \dots, a_{1\ell}a_{\ell 1}, \dots, a_{\ell\ell}a_{\ell\ell}) \\ &= \frac{1}{\prod_{ij} \gamma_{ij}!} \sum_{a_{11} \in \mathfrak{S}_{\gamma_{11}}} \sum_{a_{12} \in \mathfrak{S}_{\gamma_{12}}} \cdots \sum_{a_{\ell\ell} \in \mathfrak{S}_{\gamma_{\ell\ell}}} \prod_{ij} \chi_{\lambda_{ij}}(a_{ji}a_{ij}). \end{aligned}$$

Since  $\frac{1}{\gamma_{ii}!} \sum_{a_{ii}} \chi_{\lambda_{ii}}(a_{ii}^2) = \text{FS}_2(V(\lambda_{ii})) = 1$ , the right-hand side equals

$$\begin{aligned} & \prod_{i < j} \frac{1}{(\gamma_{ij}!)^2} \sum_{a, a' \in \mathfrak{S}_{\gamma_{ij}}} \chi_{\lambda_{ij}}(a'a) \chi_{\lambda_{ji}}(aa') \\ &= \prod_{i < j} \frac{1}{\gamma_{ij}!} \sum_{a \in \mathfrak{S}_{\gamma_{ij}}} \chi_{\lambda_{ij}}(a) \overline{\chi_{\lambda_{ji}}(a)} \\ &= \prod_{i < j} \delta_{\lambda_{ij}, \lambda_{ji}} = \delta_{\Lambda, \Lambda^\top}, \end{aligned}$$

where the first equality follows from the fact that  $\chi_{\lambda_{ji}}$  is a real-valued class function. The second equality follows from the orthogonality relation of irreducible characters.  $\square$

**Theorem 5.3** *Let  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  be a sequence of positive integers such that  $\sum_i \alpha_i = n$  and let  $\mathfrak{S}_\alpha$  be the corresponding Young subgroup of  $\mathfrak{S}_n$ . Then, for each  $b \in \mathfrak{S}_n$ , we have*

$$|\{a \in b \mathfrak{S}_\alpha \mid a^2 = 1\}| = \begin{cases} \left( \prod_{i < j} \gamma_{ij}! \right) \prod_i |\text{STab}(\gamma_{ii})| & (F^\top = F) \\ 0 & (F^\top \neq F), \end{cases} \quad (5.3)$$

where  $F = [\gamma_{ij}]$ ,  $\gamma_{ij} = |A_i \cap bA_j|$  and  $A_i$  is as in (5.1).

*Proof.* By Theorem 3.4 and Theorem 5.2, the left-hand side of (5.3) equals

$$\begin{aligned} & \sum_{\Lambda = [\lambda_{ij}] \in \mathcal{P}(F)} \delta_{F, F^\top} \delta_{\Lambda, \Lambda^\top} \dim V(\lambda_{11}) \dim V(\lambda_{12}) \cdots \dim V(\lambda_{\ell\ell}) \\ &= \delta_{F, F^\top} \left( \prod_i \sum_{\lambda_{ii} \in \mathcal{P}(\gamma_{ii})} |\text{STab}(\lambda_{ii})| \right) \left( \prod_{i < j} \sum_{\lambda_{ij} \in \mathcal{P}(\gamma_{ij})} |\text{STab}(\lambda_{ij})|^2 \right) \\ &= \delta_{F, F^\top} \left( \prod_i |\text{STab}(\gamma_{ii})| \right) \left( \prod_{i < j} \gamma_{ij}! \right) \end{aligned}$$

as desired.  $\square$

*Example.* (1) Suppose  $\alpha = (m, m')$ , where  $m' = n - m$ . Since  $F = [\gamma_{ij}] \in M_\alpha$  satisfies  $\gamma_{11} + \gamma_{12} = m = \gamma_{11} + \gamma_{21}$ , it is a symmetric matrix of the form

$$\begin{bmatrix} m - k & k \\ k & m' - k \end{bmatrix}.$$

Hence each orbital of  $X \cong \mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{m'})$  is symmetric. When  $\Gamma$  corresponds to  $G(A, bA)$ ,  $k$  equals  $|A_1 \cap bA_2| = |[m] \setminus b[m]|$ . Hence the number of involutions in  $b(\mathfrak{S}_m \times \mathfrak{S}_{m'})$  is given by (1.4). Since  $|H \cap bHb^{-1}| = |G_{A,bA}| = (m-k)!(k!)^2(m'-k)!$ , (1.5) follows from (3.7).

(2) Suppose  $\alpha = (m, 1^{n-m})$ , so that  $\mathfrak{S}_\alpha \cong \mathfrak{S}_m$  and  $A = ([m], \{m+1\}, \dots, \{n\})$ . Then,  $\gamma_{ij} = 0, 1$  unless  $(i, j) = (1, 1)$ . Assume  $\Gamma$  is symmetric. Since  $|\mathcal{P}(0)| = |\mathcal{P}(1)| = 1$ , every matrix  $A \in \mathcal{P}(\Gamma)$  is necessarily symmetric. Hence  $\text{FS}_2(\mathcal{I}_{A,B}(V)) = 1$  for each simple  $G_{AB}$ -module  $V$ . We note that this result gives a characterization of “null indicator double coset” of Schauenburg [19] Theorem 4.2. Also, we note that this gives a generalization of computations of Frobenius-Schur indicators of Hopf algebra representations given by Kashina, G. Mason, S. Montgomery [11], Jedwab, S. Montgomery [8] and Timmer [20] (cf. [19]). Since  $\gamma_{11} = m - |[m] \setminus b[m]|$ , we obtain (1.3).

## 6 Symmetric groups II

Let  $G = \mathfrak{S}_n$  be the symmetric group of  $n$ -letters. As usual, we identify the two-point stabilizer  $G_{n,n-1}$  with  $\mathfrak{S}_{n-2}$ .

**Theorem 6.1** *For each  $\mathbb{C}\mathfrak{S}_{n-2}$ -module  $V$ ,*

$$\text{FS}_r(\mathcal{I}_{n,n-1}(V)) = \sum_{2 \leq s \leq n; s|r} \text{FS}_r(V|_{\mathfrak{S}_{n-s}}), \quad (6.1)$$

where  $V|_{\mathfrak{S}_{n-s}}$  denotes the restriction of  $V$  to  $\mathfrak{S}_{n-s}$ . Here, for convenience, we set  $\text{FS}_r(V|_{\mathfrak{S}_0}) = \dim V$ .

*Proof.* Let  $a$  be an element of  $G[x, y; r]$ , where  $G[x, y; r]$  is as in Theorem 3.2,  $i_1 = x := n$  and  $i_2 = y := n-1$ . Let  $s \geq 2$  be the smallest integer such that  $a^s x = x$ . It is easy to see that  $s$  divides  $r$  and that  $s$  agrees with the size of the

orbit  $\langle a \rangle x$ . Hence we have the following decomposition:

$$G[x, y; r] = \coprod_{2 \leq s \leq n; s|r} G_s[x, y; r], \quad (6.2)$$

$$G_s[x, y; r] := \{a \in G[x, y; r] \mid |\langle a \rangle x| = s\}. \quad (6.3)$$

Suppose that  $a$  belongs to  $G_s[x, y; r]$ . We define integers  $i_3, i_4, \dots, i_s \in [n-2]$  by  $i_3 = a^2 x, \dots, i_s = a^{s-1} x$ . Then, we have  $h := a(i_1, i_2, \dots, i_s)^{-1} \in \mathfrak{S}([n] \setminus \{i_1, \dots, i_s\})$ , that is,  $h$  fixes each element of  $\{i_1, \dots, i_s\}$ . Conversely, if  $i_3, \dots, i_s$  are distinct elements of  $[n-2]$  and  $h$  is an element of  $\mathfrak{S}([n] \setminus \{i_1, \dots, i_s\})$ ,  $a = h(i_1, \dots, i_s)$  gives an element of  $G_s[x, y; r]$ . Therefore, we have

$$G_s[x, y; r] = \coprod_{i_3, \dots, i_s} \{h(i_1, \dots, i_s) \mid h \in \mathfrak{S}([n] \setminus \{i_1, \dots, i_s\})\}, \quad (6.4)$$

where  $i_3, \dots, i_s$  run over distinct elements of  $[n-2]$ . By (3.4), this implies

$$\begin{aligned} & \text{FS}_r(\mathcal{I}_{n,n-1}(V)) \\ &= \frac{1}{(n-2)!} \sum_{2 \leq s \leq n; s|r} \sum_{i_3, \dots, i_s} \sum_h \text{Tr}_V((h(i_1, \dots, i_s))^{-r}). \end{aligned}$$

where  $h$  runs over  $\mathfrak{S}([n] \setminus \{i_1, \dots, i_s\})$ . Since  $h$  commutes with the permutation  $(i_1, \dots, i_s) \in \mathfrak{S}(\{i_1, \dots, i_s\})$ , the right-hand side of the above equality is

$$\begin{aligned} & \frac{1}{(n-2)!} \sum_{2 \leq s \leq n; s|r} \sum_{i_3, \dots, i_s} \sum_h \text{Tr}_V(h^{-r}) \\ &= \frac{1}{(n-2)!} \sum_{2 \leq s \leq n; s|r} \frac{(n-2)!}{(n-s)!} \sum_{h \in \mathfrak{S}_{n-s}} \text{Tr}_V(h^{-r}) \\ &= \sum_{2 \leq s \leq n; s|r} \text{FS}_r(V|_{\mathfrak{S}_{n-s}}). \end{aligned}$$

□

**Theorem 6.2** (1) *We have*

$$R_{\mathfrak{S}_n, (n-1, n) \mathfrak{S}_{n-1}}^r|_{\mathfrak{S}_{n-2}} = \sum_{2 \leq s \leq n; s|r} \text{Ind}_{\mathfrak{S}_{n-s}}^{\mathfrak{S}_{n-2}}(R_{\mathfrak{S}_{n-s}}^r), \quad (6.5)$$

where  $\text{Ind}_{\mathfrak{S}_{n-s}}^{\mathfrak{S}_{n-2}}(R_{\mathfrak{S}_{n-s}}^r)$  denotes the induced class function of  $R_{\mathfrak{S}_{n-s}}^r$  on  $\mathfrak{S}_{n-2}$ .

(2) *The class function  $R_{\mathfrak{S}_n, (n-1, n) \mathfrak{S}_{n-1}}^r|_{\mathfrak{S}_{n-2}}$  is a character of a certain representation of  $\mathfrak{S}_{n-2}$ .*

*Proof.* Let  $H$  be a finite group. By (1.2) and the orthogonal relation of the irreducible characters, we have  $\text{FS}_r(W) = (R_H^r | \chi_W)_H$  for each simple  $\mathbb{C}H$ -module  $W$ . Since  $\text{FS}_r$  is additive, this relation also holds for every finite-dimensional  $\mathbb{C}H$ -module  $W$ . Suppose  $H$  is a subgroup of a finite group  $G$ . By Frobenius reciprocity, we have  $\text{FS}_r(V|_H) = (\text{Ind}_H^G(R_H^r) | \chi_V)_G$  for each finite-dimensional  $\mathbb{C}G$ -module  $V$ . Applying this equality to  $G = \mathfrak{S}_{n-2}$  and  $H = \mathfrak{S}_{n-s}$  together with (6.1), we find that

$$\text{FS}_r(\mathcal{I}_{n,n-1}(V)) = \sum_{2 \leq s \leq n; s|r} (\text{Ind}_{\mathfrak{S}_{n-s}}^{\mathfrak{S}_{n-2}}(R_{\mathfrak{S}_{n-s}}^r) | \chi_V)_{\mathfrak{S}_{n-2}}.$$

Hence by Theorem 3.4, we get

$$\begin{aligned} R_{\mathfrak{S}_n, (n-1, n) \mathfrak{S}_{n-1}}^r |_{\mathfrak{S}_{n-2}} &= \sum_{\mu \in \mathcal{P}_{n-2}} \sum_{2 \leq s \leq n; s|r} (\text{Ind}_{\mathfrak{S}_{n-s}}^{\mathfrak{S}_{n-2}}(R_{\mathfrak{S}_{n-s}}^r) | \chi_\mu)_{\mathfrak{S}_{n-2}} \chi_\mu \\ &= \sum_{2 \leq s \leq n; s|r} \text{Ind}_{\mathfrak{S}_{n-s}}^{\mathfrak{S}_{n-2}}(R_{\mathfrak{S}_{n-s}}^r), \end{aligned}$$

where the last equality follows from the fact that  $\{\chi_\mu | \mu \in \mathcal{P}_{n-2}\}$  is an orthonormal basis of the space of class functions of  $\mathfrak{S}_{n-2}$ . Thus we get Part (1). Part (2) follows immediately from Part (1) and [16].  $\square$

**Corollary 6.3** (1) For each  $b \in \mathfrak{S}_n \setminus \mathfrak{S}_{n-1}$ ,

$$|\{a \in b\mathfrak{S}_{n-1} | a^r = 1\}| = \sum_{2 \leq s \leq n; s|r} \frac{(n-2)!}{(n-s)!} |\{a \in \mathfrak{S}_{n-s} | a^r = 1\}|. \quad (6.6)$$

(2) (cf. [3]) The root number  $R_{\mathfrak{S}_n}^r(1)$  satisfies the recurrence relation

$$R_{\mathfrak{S}_n}^r(1) = \sum_{1 \leq s \leq n; s|r} \frac{(n-1)!}{(n-s)!} R_{\mathfrak{S}_{n-s}}^r(1). \quad (6.7)$$

*Proof.* Since the induced class function  $\text{Ind}_H^G(f)$  satisfies  $\text{Ind}_H^G(f)(1) = \frac{|G|}{|H|} f(1)$ , (6.6) immediately follows from (6.5) when  $b = (n-1, n)$ . On the other hand, since  $\mathfrak{S}_{n-1} \setminus \mathfrak{S}_n / \mathfrak{S}_{n-1} = \{\mathfrak{S}_{n-1}, \mathfrak{S}_{n-1}(n-1, n)\mathfrak{S}_{n-1}\}$ , we have  $R_{G, b\mathfrak{S}_{n-1}}^r(1) = R_{G, (n-1, n)\mathfrak{S}_{n-1}}^r(1)$  by (3.6). This proves Part (1). Part (2) follows from Part (1) and (3.7).  $\square$



## 7 A correspondence between bilinear pairings

As well as the group case, the second Frobenius-Schur indicator of an  $\mathcal{F}(G, X)$ -module  $M$  has a close connection to invariant bilinear forms on  $M$ . In this section, we show it by giving a correspondence between invariant bilinear forms on  $\mathcal{F}(G, X)$ -modules and certain bilinear pairings on  $\mathbb{C}G_{xy}$ -modules.

Let  $G$  be a finite group and let  $K$  be its subgroup. Let  $t$  be an element of  $G$  such that  $t^{-1}Kt = K$  and  $t^2 \in K$ . For a  $\mathbb{C}K$ -module  $V$ , we denote by  ${}^tV = \{{}^tv \mid v \in V\}$  a copy of  $V$  with  $\mathbb{C}K$ -action given by  $k{}^tv := {}^t(t^{-1}ktv)$ . Let  $B : V \times {}^tV \rightarrow \mathbb{C}$  be a bilinear pairing. We say that  $B$  is  $K$ -invariant if  $B(kv, {}^tw) = B(v, k^{-1}{}^tw)$  for each  $k \in K$  and  $v, w \in V$ . We denote by  $\mathcal{B}(V, t)$  the set of  $K$ -invariant bilinear pairings  $B : V \times {}^tV \rightarrow \mathbb{C}$ . For  $B \in \mathcal{B}(V, t)$ , we set  $B^\top(v, {}^tw) := B(t^2w, {}^tv)$ . Since

$$\begin{aligned} B^\top(kv, {}^tw) &= B(t^2w, tkt^{-1}{}^tv) = B(tk^{-1}tw, {}^tv) = B^\top(v, k^{-1}{}^tw), \\ (B^\top)^\top(v, {}^tw) &= B(t^2v, t^2{}^tw) = B(v, {}^tw), \end{aligned}$$

we have  $B^\top \in \mathcal{B}(V, t)$  and  $(B^\top)^\top = B$ . Similarly to [10], we have the following result.

**Proposition 7.1** *Let  $V$  be a simple  $\mathbb{C}K$ -module. Then  $\text{FS}_2(V, t) \in \{0, \pm 1\}$  and  $\dim \mathcal{B}(V, t) \leq 1$ . Moreover, we have*

$$\text{FS}_2(V, t) = \begin{cases} 1 & \dim \mathcal{B}(V, t)_+ = 1 \\ -1 & \dim \mathcal{B}(V, t)_- = 1 \\ 0 & \dim \mathcal{B}(V, t) = 0, \end{cases} \quad (7.1)$$

where  $\mathcal{B}(V, t)_\pm := \{B \in \mathcal{B}(V, t) \mid B^\top = \pm B\}$ .

Let  $M$  be a  $\mathcal{F}(G, X)$ -module and let  $C : M \times M \rightarrow \mathbb{C}$  be a bilinear form on  $M$ . We say that  $C$  is  $\mathcal{F}(G, X)$ -invariant if it is  $G$ -invariant and satisfies  $C(e_y^x \xi, \eta) = C(\xi, e_x^y \eta)$  for each  $x, y \in X$  and  $\xi, \eta \in M$ . We denote by  $\mathcal{B}(M)$  the set of  $\mathcal{F}(G, X)$ -invariant bilinear forms. For  $C \in \mathcal{B}(M)$ , we define  $C^\top \in \mathcal{B}(M)$  by  $C^\top(\xi, \eta) = C(\eta, \xi)$ .

**Theorem 7.2** *Let  $\Omega = G(x, y)$  be a symmetric orbital and let  $t$  be an element of  $G$  such that  $t(x, y) = (y, x)$ .*

(1) *For each  $\mathbb{C}G_{xy}$ -module  $V$ , we have a bijective correspondence*

$$\text{Res}: \mathcal{B}(\mathcal{I}_{x,y}(V)) \cong \mathcal{B}(V, t)$$

*given by  $\text{Res}(C)(v, {}^t w) = C(1 \otimes v, t \otimes w)$ . The inverse  $\text{Ind}$  of  $\text{Res}$  is given by  $\text{Ind}(B)(a \otimes v, b \otimes w) = \sum_{k \in K} \delta_{akt, b} B(v, k {}^t w)$ , where  $K = G_{xy}$ .*

(2) *For each  $C \in \mathcal{B}(\mathcal{I}_{x,y}(V))$ , we have  $\text{Res}(C^\top) = \text{Res}(C)^\top$ .*

(3) *A pairing  $B \in \mathcal{B}(V, t)$  is non-degenerate if and only if  $\text{Ind}(B)$  is non-degenerate.*

*Proof.* It is straightforward to verify that  $\text{Res}$  and  $\text{Ind}$  give well-defined maps between  $\mathcal{B}(\mathcal{I}_{x,y}(V))$  and  $\mathcal{B}(V, t)$ . Also, it is easy to verify that  $\text{Res} \circ \text{Ind} = \text{id}$ . Hence, to show Part (1), it suffices to prove that

$$\text{Ind}(\text{Res}(C))(a \otimes v, b \otimes w) = C(a \otimes v, b \otimes w) \quad (7.2)$$

for each  $a, b \in G$  and  $v, w \in V$ . By  $\mathcal{F}(G, X)$ -invariance, the left and right-hand sides of (7.2) is zero unless  $a(x, y) = b(y, x)$ . Suppose  $a(x, y) = b(y, x)$ . Since  $k := a^{-1}bt^{-1} \in K$ , the left-hand side of (7.2) is

$$\text{Res}(C)(v, k {}^t w) = C(1 \otimes v, t \otimes t^{-1}ktw) = C(1 \otimes v, kt \otimes w) = C(a \otimes v, b \otimes w).$$

This proves Part (1). Part (2) follows from  $t^2 \in K$ . Suppose that  $C = \text{Ind}(B) \in \mathcal{B}(\mathcal{I}_{x,y}(V))$  is non-degenerate and that  $v \in V$  satisfies  $B(v, {}^t w) = 0$  for every  $w \in V$ . To prove the non-degeneracy of  $B$ , it suffices to show that  $C(1 \otimes v, b \otimes w) = 0$  for each  $b \in G$  and  $w \in V$ . By  $\mathcal{F}(G, X)$ -invariance, we may assume  $b(x, y) = (y, x)$ , or  $b = tk$  for some  $k \in K$ . Then, we obtain  $C(1 \otimes v, b \otimes w) = B(v, {}^t(kw)) = 0$  and prove the non-degeneracy of  $B$ . Conversely, suppose that  $B$  is non-degenerate and that  $m \in e_w^z \mathcal{I}_{x,y}(V)$  satisfies  $C(m, n) = 0$  for every  $n \in e_z^w \mathcal{I}_{x,y}(V)$ . Let  $a$  be an arbitrary element of  $G$  such that  $a(x, y) = (z, w)$ . Then, we have that  $m = a \otimes v$  and  $n = at \otimes w$  for some  $v, w \in V$ , and that  $B(v, {}^t w) = C(m, n) = 0$ . Hence, the non-degeneracy of  $B$  implies  $m = 0$ . Since  $C$  is  $\mathcal{F}(G, X)$ -invariant, this proves the non-degeneracy of  $C$ .  $\square$

**Corollary 7.3** *Let  $M$  be a simple  $\mathcal{F}(G, X)$ -module. Then  $\text{FS}_2(M) \in \{0, \pm 1\}$  and  $\dim \mathcal{B}(M) \leq 1$ . Moreover, we have*

$$\text{FS}_2(M) = \begin{cases} 1 & \dim \mathcal{B}(M)_+ = 1 \\ -1 & \dim \mathcal{B}(M)_- = 1 \\ 0 & \dim \mathcal{B}(M) = 0, \end{cases} \quad (7.3)$$

where  $\mathcal{B}(M)_\pm := \{C \in \mathcal{B}(M) \mid C^\top = \pm C\}$ .

*Proof.* Suppose  $M$  is of type  $\Omega$ . When  $\Omega$  is symmetric, the assertion follows immediately from Theorem 4.2 (2), Proposition 7.1 and Theorem 7.2. When  $\Omega$  is not symmetric,  $\mathcal{B}(M) = 0$  by the definition of  $\mathcal{F}(G, X)$ -invariance. Hence the assertion follows from Theorem 4.2 (1).  $\square$

**Proposition 7.4** *Let  $x, y$  and  $b$  be as in Theorem 3.4 and let  $\Omega$  be  $G(x, y)$ . Then the following conditions are equivalent:*

- (1)  $\Omega^\top = \Omega$ .
- (2)  $R_{G, bH}^2 \neq 0$ .
- (3)  $\text{FS}_2(M) \neq 0$  for some  $\mathcal{F}(G, X)$ -module  $M$  of type  $\Omega$ .

*Proof.* The equivalence of (2) and (3) follows from Theorem 3.4 and the linear independence of the characters. Since the unit  $\mathbb{C}G_{xy}$ -module  $\mathbb{C}$  satisfies  $\dim \mathcal{B}(\mathbb{C}, t)_+ = 1$ , the equivalence of (1) and (3) follows from Theorem 7.2 (1), Corollary 7.3 and Theorem 4.2 (1).  $\square$

## 8 Frobenius-Schur indicators of Ng and Schauenburg

In [15], Ng and Schauenburg have defined higher Frobenius-Schur indicators  $\nu_r(M)$  for each pivotal tensor category  $\mathcal{C}$  and its object  $M$ . In this section, we verify that  $\text{FS}_r$  coincides with  $\nu_r$  when  $\mathcal{C}$  is the category  $\mathcal{F}\mathbf{mod}$  of finite-dimensional left  $\mathcal{F}$ -modules, where  $\mathcal{F} = \mathcal{F}(G, X)$  for some  $(G, X)$ . We refer to [5] for terminology for tensor categories. To begin with, we give an explicit

description of operations on  $\mathcal{F}\mathbf{mod}$ . For each  $M, N \in \text{ob}\mathcal{F}\mathbf{mod}$ , let  $M \overline{\otimes} N$  be a subspace  $M \otimes N$  defined by  $M \overline{\otimes} N := \Delta(1)(M \otimes N) = \bigoplus_{z \in X} e_z M \otimes e^z N$ , where  $e_y = \sum_x e_y^x$  and  $e^x = \sum_y e_y^x$ . Then  $M \overline{\otimes} N$  becomes an  $\mathcal{F}$ -module via

$$e_y^x a \sum_z e_z m \otimes e^z n = \sum_z e_z e^x a m \otimes e^z e_y a n \quad (a \in G, x, y, z \in X, m \in M, n \in N).$$

The linear span  $\mathbf{1} := \mathbb{C}X$  of  $X$  becomes an  $\mathcal{F}$ -module via  $e_y^x a \otimes z \mapsto \delta_{x,az} \delta_{y,az} a z$ . Moreover it becomes a unit object with respect to  $\overline{\otimes}$  via

$$\begin{aligned} M &\cong \mathbf{1} \overline{\otimes} M = \bigoplus_x \mathbb{C}x \otimes e^x M; \quad m \mapsto \sum_x x \otimes e^x m, \\ M &\cong M \overline{\otimes} \mathbf{1} = \bigoplus_x e_x M \otimes \mathbb{C}x; \quad m \mapsto \sum_x e_x m \otimes x. \end{aligned} \quad (8.1)$$

The linear dual  $M^*$  of  $M$  has an  $\mathcal{F}$ -module structure, which is determined by

$$\langle e_y^x a m', m \rangle = \langle m', a^{-1} e_x^y m \rangle \quad (a \in G, x, y \in X, m' \in M^*, m \in M).$$

The module  $M^*$  becomes a left dual object of  $M$  via

$$\begin{aligned} ev: M^* \overline{\otimes} M &\rightarrow \mathbf{1}; \quad \sum_x e_x m' \otimes e^x m \mapsto \sum_y \langle e^y m', m \rangle y, \\ coev: \mathbf{1} &\rightarrow M \overline{\otimes} M^*; \quad x \mapsto \sum_i e^x m_i \otimes m^i, \end{aligned} \quad (8.2)$$

where  $\{m_i\}$  denotes a basis of  $M$  and  $\{m^i\}$  denotes its dual basis. The canonical linear isomorphism  $j_M: M \cong M^{**}$  becomes an isomorphism of  $\mathcal{F}$ -modules. Hence  $\mathcal{C} = \mathcal{F}\mathbf{mod}$  becomes a pivotal tensor category.

For each  $M, N \in \text{ob}\mathcal{C}$ , we define linear maps  $A_{M,N}: \mathcal{C}(\mathbf{1}, M \overline{\otimes} N) \rightarrow \mathcal{C}(M^*, N)$ ,  $T_{M,N}: \mathcal{C}(M^*, N) \rightarrow \mathcal{C}(N^*, M)$ ,  $E_{M,N}: \mathcal{C}(\mathbf{1}, M \overline{\otimes} N) \rightarrow \mathcal{C}(\mathbf{1}, N \overline{\otimes} M)$  by

$$\begin{aligned} A_{M,N}(f): M^* &\cong M^* \overline{\otimes} \mathbf{1} \xrightarrow{\text{id} \overline{\otimes} f} M^* \overline{\otimes} M \overline{\otimes} N \xrightarrow{ev \overline{\otimes} \text{id}} \mathbf{1} \overline{\otimes} N \cong N. \\ T_{M,N}(g) &= j_M^{-1} \circ g^*, \quad E_{M,N}(f) = (A_{N,M}^{-1} \circ T_{M,N} \circ A_{M,N})(f), \end{aligned}$$

respectively, where  $f \in \mathcal{C}(\mathbf{1}, M \overline{\otimes} N)$  and  $g \in \mathcal{C}(M^*, N)$ .

Then, the  $r$ -th indicator  $\nu_r(M)$  of  $M \in \text{ob}\mathcal{F}\mathbf{mod}$  is defined by  $\nu_r(M) := \text{Tr}(E_{M, M \overline{\otimes}^{r-1}})$ .

Let  $M$  be a finite-dimensional vector space. We say that  $M$  is an  $\mathcal{F}$ -space if it is equipped with an associative action  $\mathcal{F} \otimes M \rightarrow M$ , that is, the corresponding linear map  $\pi_M : \mathcal{F} \rightarrow \text{End}(M)$  satisfies  $\pi_M(\alpha\beta) = \pi_M(\alpha)\pi_M(\beta)$  ( $\alpha, \beta \in \mathcal{F}$ ). Let  $N$  be another  $\mathcal{F}$ -space. Then  $M \otimes N$  becomes an  $\mathcal{F}$ -space via  $\pi_{M \otimes N}(\alpha) = (\pi_M \otimes \pi_N)(\Delta(\alpha))$  ( $\alpha \in \mathcal{F}$ ). For each  $\mathcal{F}$ -space  $M$ , we set  $\overline{M} := \pi_M(1)M$  and  $M^{\mathcal{F}} := \pi_M(f)M$ . Then  $\overline{M}$  becomes an  $\mathcal{F}$ -module.

**Lemma 8.1** *Let  $M$  and  $N$  be  $\mathcal{F}$ -spaces.*

(1) *We have  $\overline{\overline{M} \otimes N} = \overline{M \otimes N} = \overline{M \otimes \overline{N}}$ .*

(2) *Let  $\varepsilon^L$  be as in (3.2). then, we have*

$$M^{\mathcal{F}} = \{m \in M \mid \alpha m = \varepsilon^L(\alpha)m \quad (\alpha \in \mathcal{F})\}. \quad (8.3)$$

(3) *The twist map  $\text{tw}_{M,N} : M \otimes N \rightarrow N \otimes M$ ;  $m \otimes n \mapsto n \otimes m$  satisfies  $\text{tw}_{M,N} \circ \pi_{M \otimes N}(f) = \pi_{N \otimes M}(f) \circ \text{tw}_{M,N}$ . In particular, it gives a linear isomorphism  $(M \otimes N)^{\mathcal{F}} \cong (N \otimes M)^{\mathcal{F}}$ .*

*Proof.* Part (1) is obvious. Let  $N$  be the right-hand side of (8.3). By (3.1), we have  $M^{\mathcal{F}} \subseteq N$ . On the other hand, since  $\varepsilon^L(f) = 1$ , we have  $n = \pi_M(f)n \in M^{\mathcal{F}}$  for each  $n \in N$ . Part (3) follows from

$$(\text{tw}_{M,N} \circ \pi_{M \otimes N}(f))(m \otimes n) = \frac{1}{|G|} \sum_{x,y,a} e_x^y a n \otimes e_y^x a m = (\pi_{N \otimes M}(f) \circ \text{tw}_{M,N})(m \otimes n).$$

□

For each  $\mathcal{F}$ -space  $M$ , there exists a linear isomorphism  $\iota_M : M^{\mathcal{F}} \cong \mathcal{C}(1, \overline{M})$  such that  $\iota_M(m)(x) = e^x m$  for each  $m \in M^{\mathcal{F}}$  and  $x \in X$ . The inverse of  $\iota_M$  is given by  $\iota_M^{-1}(f) = \sum_{x \in X} f(x)$ .

**Lemma 8.2** *For each  $\mathcal{F}$ -spaces  $M$  and  $N$ , the diagram*

$$\begin{array}{ccc} (M \otimes N)^{\mathcal{F}} & \xrightarrow{\text{tw}_{M,N}} & (N \otimes M)^{\mathcal{F}} \\ \iota_{M \otimes N} \downarrow & & \downarrow \iota_{N \otimes M} \\ \mathcal{C}(1, \overline{M \otimes N}) & \xrightarrow{E_{\overline{M}, \overline{N}}} & \mathcal{C}(1, \overline{N \otimes M}) \end{array}$$

*is commutative.*

*Proof.* Let  $\sum_i m_i \otimes n_i$  be an element of  $(M \otimes N)^{\mathcal{F}}$ . Set  $g_1 = (A_{M,N} \circ \iota_{M \otimes N})(\sum_i m_i \otimes n_i)$  and  $g_2 = (A_{N,M} \circ \iota_{N \otimes M})(\sum_i n_i \otimes m_i)$ . It is straightforward to verify that  $g_1(m') = \sum_i \langle m', m_i \rangle n_i$  for each  $m' \in \overline{M}^*$ . Hence

$$\langle m', T_{M,N}(g_1)(n') \rangle = \sum_i \langle n', n_i \rangle \langle m', m_i \rangle = \langle m', g_2(n') \rangle$$

for each  $m' \in \overline{M}^*$  and  $n' \in \overline{N}^*$ . This proves the assertion.  $\square$

**Proposition 8.3** *For each  $M \in \text{ob}_{\mathcal{F}\text{-mod}}$ , we have  $\text{FS}_r(M) = \nu_r(M)$ .*

*Proof.* Applying Lemma 8.2 to  $N = M^{\otimes r-1}$  and using Lemma 8.1 (1), we obtain  $\nu_r(M) = \text{Tr}_{(M^{\otimes r})^{\mathcal{F}}}(\text{tw}_{M, M^{\otimes r-1}})$ . Since  $\int$  is an idempotent, this equals to  $\text{Tr}(\pi_{M^{\otimes r}}(\int) \circ \text{tw}_{M, M^{\otimes r-1}}) = \text{Tr}(\pi_M^{\otimes r}(\Delta^{(r)}(\int)) \circ \text{tw}_{M, M^{\otimes r-1}})$  by Lemma 8.1 (3). Hence, the assertion follows from the formula  $\text{Tr}((f_1 \otimes \cdots \otimes f_r) \circ \text{tw}_{M, M^{\otimes r-1}}) = \text{Tr}(f_1 \circ \cdots \circ f_r)$ , which holds for each  $f_1, \dots, f_r \in \text{End}(M)$ .  $\square$

## References

- [1] N. Andruskiewitsch and S. Natale, Tensor categories attached to double groupoids, *Adv. Math.* 200 (2006) 539-583.
- [2] G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras, I. Integral theory and  $C^*$ -structure, *J. Algebra* 221 (1999) 385-438.
- [3] S. Chowla, I. N. Herstein, W. R. Scott, The solutions of  $x^d = 1$  in symmetric groups, *Norske Vid. Selsk. Forh.*, Trondheim 25 (1952) 29-31.
- [4] C. W. Curtis, L. Reiner, *Methods of representation theory, I*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1981.
- [5] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, American Mathematical Society, 2015.
- [6] J. Fuchs, C. Ganchev, K. Szlachányi and P. Vescernyés,  $S_4$ -symmetries of  $6j$  symbols and Frobenius-Schur indicators in rigid monoidal  $C^*$ -categories, *J. Math. Phys.* 40 (1999) 408-426.

- [7] T. Hayashi, Quantum group symmetry of partition functions of IRF models and its application to Jones' index theory, *Commun. Math. Phys.* 157 (1993) 331-345.
- [8] A. Jedwab, S. Montgomery, Representations of some Hopf algebras associated to the symmetric group  $S_n$ , *Algebr. Represent. Theory* 12 (2009) 1-17.
- [9] A. Kerber, *Applied finite group actions*, Springer Science & Business Media, 2013.
- [10] N. Kawanaka, H. Matsuyama, A twisted version of the Frobenius-Schur indicator and multiplicity-free permutation representations, *Hokkaido Math. J.* 19 (1990) 495-508.
- [11] Y. Kashina, G. Mason, S. Montgomery, Computing the Frobenius-Schur indicator for abelian extensions of Hopf algebras, *J. Algebra* 251 (2002) 888-913.
- [12] G. Lusztig, Leading coefficients of character values of Hecke algebras, in: *Proc. Symp. Pure Math.* 1987, pp. 235-262.
- [13] J. M. Mombelli, S. Natale, Tensor categories and vacant double groupoids, *arXiv:0509052*, 2005.
- [14] S. Natale, Frobenius-Schur indicators for a class of fusion categories, *Pacific J. Math.* 221 (2005) 353-377.
- [15] S. H. Ng, P. Schauenburg, Higher Frobenius-Schur indicators for pivotal categories, in: *Hopf algebras and generalizations*, in: *Contemp. Math.*, vol. 441, Amer. Math. Soc., Providence, RI, 2007, pp. 63-90.
- [16] T. Scharf, Die Wurzelanzahlfunktion in symmetrischen Gruppen, *J. Algebra* 139 (1991), 446-457.

- [17] P. Schauenburg, Computing higher Frobenius-Schur indicators in fusion categories constructed from inclusions of finite groups, *Pacific J. Math.* 280 (2015) 177-201.
- [18] P. Schauenburg, A higher Frobenius-Schur indicator formula for group-theoretical fusion categories, *Comm. Math. Phys.* 340 (2015) 833-849.
- [19] P. Schauenburg, Frobenius-Schur indicators for some fusion categories associated to symmetric and alternating groups, *Algebras and Representation Theory* (2015) 1-12.
- [20] J. Timmer, Indicators of bismash products from exact symmetric group factorizations, *arXiv:1412.4725*, 2014.